FLUID MECHANICS NOTES

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1.1 Review

There are a few concepts that need to be reviewed to aid in understanding the text.

1.1.1 Density

$$\rho = \frac{mass}{volume} \quad Units: \frac{kg}{m^3}; \frac{g}{cm^3}: \frac{lb_m}{ft^3} \tag{1.1}$$

Density is an extremely important property of matter. The density of a material can be considered continuous except at the molecular level. Density can also be thought of as the constant that relates mass to volume. This makes it easy to convert between the two.

1.1.2 Specific Gravity

$$SG = \frac{\rho}{\rho_{ref}} \quad Where\ \rho_{ref} = 1000\frac{kg}{m^3}; 1\frac{g}{cm^3}; 62.4\frac{lb_m}{ft^3}; 8.33\frac{lb_m}{gal} \tag{1.2}$$

Specific gravity is used instead of density to tabulate data for different materials. Using the specific gravity, the density in any set of units may be found by picking the reference density in the desired units. Note: When the reference density is expressed in $\frac{g}{cm^3}$, the density and specific gravity have the same numerical value.

1.1.3 Specific Weight

$$\gamma = \rho\frac{g}{g_c} \quad Units: \frac{N}{m^3}; \frac{dyne}{cm^3}; \frac{lb_f}{ft^3} \tag{1.3}$$

The specific weight (1.3) is a quantity that is used frequently in fluid mechanics. In the American Engineering Series (AES) of units, it is numerically equal to the density. The units are lb_f rather than lb_m. The choice of γ (gamma) for the symbol for specific weight was somewhat unfortunate since another important variable in fluid mechanics (strain) also uses γ. To avoid this problem, your book uses Γ for strain. This is non-standard and can cause confusion since Γ is rarely used for this purpose in the literature.

1.2 Newtonian Fluids

A fluid is defined as a material that can not support a stress or as a material that is continuously deformed by the application of a stress.

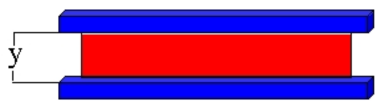


Figure 1.1: A fluid element before deformation.

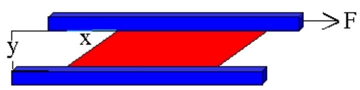


Figure 1.2: Fluid element after the application of a force acting tangentially on the top of the element.

$$shear\ stress(\tau\ or\ \sigma) = \frac{force}{area} \quad Units: \frac{N}{m^2}; \frac{dyne}{cm^2}; \frac{lb_f}{ft^2}; \frac{lb_f}{in^2} \tag{1.4}$$

$$strain\ (\gamma) = \frac{displacement}{element\ height} = \frac{x}{y} \tag{1.5}$$

Since a fluid is continuously deformed by a force acting on it, strain doesn't mean much, but the rate of strain or shear rate ($\dot{\gamma}$) is important.

$$strain\ rate\ or\ shear\ rate\ (\dot{\gamma}) = \frac{d\gamma}{dt} = \frac{d\frac{x}{y}}{dt} \quad Units: \frac{1}{time} \tag{1.6}$$

Newton first proposed that the shear stress could be related to the shear rate by

$$\tau = Constant\ \dot{\gamma}$$

The constant is termed the viscosity (μ). It is a constant of proportionality between shear stress and shear rate. Viscosity is analogous to a modulus.

$$\tau = \mu \, \dot{\gamma} \implies \mu = \frac{\tau}{\dot{\gamma}} \quad Units: \frac{F \cdot t}{area}; \frac{dyne \cdot s}{cm^2}; \frac{N \cdot s}{m^2} \; or \; Pa \cdot s \tag{1.7}$$

The unit ($\frac{dyne \cdot s}{cm^2}$) is called a *Poise*. It is more common to use *centipoise* (cp) or $0.01 Poise$. Water has a viscosity of 1 cp while honey has a viscosity of about 400 cp. It is easy to confuse *Poise* and *centipoise* when making calculations. Remember, a Poise is equal to $1, \frac{dyne \cdot s}{1cm^2}$. A $Pa \cdot s$ is equal to 0.1 Poise. Viscosity can also be expressed in $\frac{lb_f \cdot s}{ft^2}$. The viscosity of water in the AES system of units is $2.1x10^{-5}$ $\frac{lb_f \cdot s}{ft^2}$. Converting from *Poise* to $\frac{lb_f \cdot s}{ft^2}$ is accomplished by multiplying *Poise* by 1 $\frac{lb_f \cdot s}{ft^2}/478.8$ *Poise*. The equivalents for viscosity are

$$1 \, \frac{lb_f \cdot s}{ft^2} = 47.88 \, \frac{N \cdot s}{m^2} = 478.8 \; Poise = 47880 \; centipoise$$

1.3 Non-Newtonian Fluids

Fluids that exhibit a nonlinear relationship between stress and strain rate are termed non-Newtonian fluids. Many common fluids that we see everyday are non-Newtonian. Paint, peanut butter, and toothpaste are good examples. High viscosity does not always imply non-Newton behavior. Honey is viscous and Newtonian while a 5W30 motor oil is not very viscous, but it is non-Newtonian. There are several types of non-Newtonian fluids. Figure 1.3 shows several of the more common types.

1.3.1 Power Law Fluids

Power law fluids are fluids that follow the power law (Equation 1.8) over part or all of the shear rate range. These fluids are also known as pseudoplastic fluids. Where m is the consistency index (K is also used in the literature) with units of $\frac{F \cdot s^n}{area}$ and n is the power law or flow

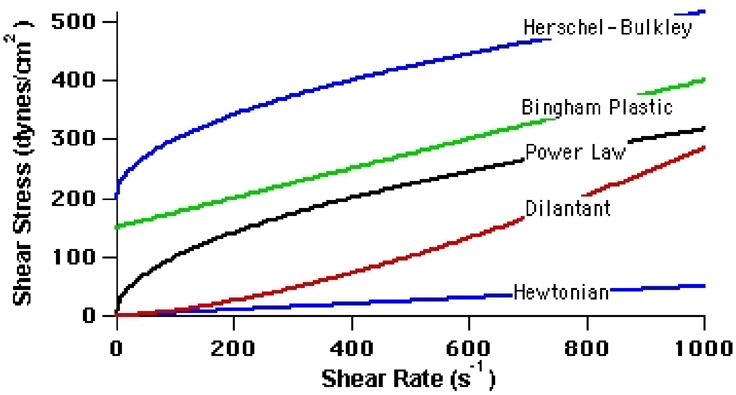


Figure 1.3: Types of non-Newtonian fluids.

behavior index. For power law fluids, n ranges from 0 to 1. While it is greater than one for dilatant fluids. The value of m or K depends upon the system of units. Viscosity of a power law fluid is obviously a function of shear rate and not constant as it is for Newtonian fluids. There are two different ways to describe the viscosity: the slope of the tangent line at any point on the curve and the slope of the line drawn from the origin to the shear rate of interest (Figure 1.4). The latter is the preferred method and is termed the apparent viscosity and given the symbol (η_a). This is short for the apparent Newtonian viscosity because it is the viscosity that a Newton fluid would have if the line is based on a single point measurement. Because the viscosity is a function of shear rate it is necessary to specify the shear rate at which the viscosity is reported $(\eta_a(\dot{\gamma}))$.

$$\tau = m\dot{\gamma}^n \tag{1.8}$$

$$\eta_a(\dot{\gamma}) = \frac{\tau}{\dot{\gamma}} = \frac{m\,\dot{\gamma}^n}{\dot{\gamma}} = \frac{m}{\dot{\gamma}^{1-n}} \tag{1.9}$$

1.3.2 Bingham Plastic

Bingham plastic fluids are fluids that exhibit a yield stress (Figure 1.3. This means that the fluid will support a stress up to a point before flow

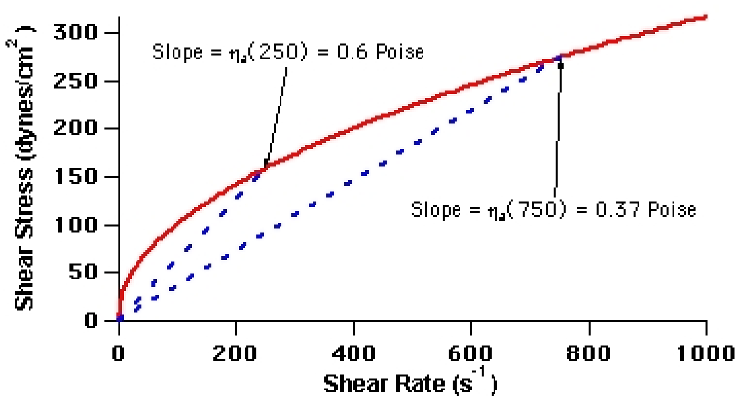


Figure 1.4: Apparent Newtonian viscosity.

begins. Good paints are Bingham plastics. The flow behavior of Bingham plastic fluids is described by

$$\tau = \tau_0 + \mu_p\,\dot{\gamma} \tag{1.10}$$

Where τ_0 is the yield stress (the stress that must be exceeded before flow begins) and μ_p is the plastic viscosity.

1.3.3 Herschel-Bulkley Fluids

The Herschel-Bulkley formulation is a generalization of the Bingham plastic equation. In the Bingham plastic model, the viscosity is constant after the yield stress is exceeded while the Herschel-Bulkley model allows for power law behavior.

$$\tau = \tau_0 + m\,\dot{\gamma}^n \tag{1.11}$$

Looking at the various parameters in the Herschel-Bulkley equation, we can make the following observations for

$$\tau = \tau_0 + m\,\dot{\gamma}^n$$

when

$$\begin{aligned}
\tau_0 = 0 \quad & \Longrightarrow \tau = m\dot{\gamma}^n && \mathit{PowerLaw} \\
\tau_0 = 0 \; and \; n = 1 \quad & \Longrightarrow \tau = \mu\dot{\gamma} && \mathit{Newtonian} \\
n = 1 \quad & \Longrightarrow \tau = \tau_0 + \mu_p\dot{\gamma} && \mathit{BinghamPlastic}
\end{aligned}$$

The Herschel-Bulkley fluid model can be reduced to the three other models and is therefore the most general of the simple fluid models.

1.3.4 Dilatant Fluids

Dilatant fluids are shear thickening fluids. This means that the viscosity increases with shear. There are few examples of dilatant fluids. Most dilatant fluids are concentrated slurries. The power law can be used to describe dilatant fluid behavior ($n > 1$).

1.3.5 Time Dependent Fluids

Time dependent fluids are fluids that increase or decrease in viscosity over time at a constant shear rate. The construction industry uses a cement slurry that thins with time at a constant pump rate because it is easier to pump and fills forms easily. There are two types of time dependent fluids: Rheopectic and Thixotropic.

Rheopectic Fluids

Rheopectic fluids increase in viscosity with time at constant shear rate. There are few examples of rheopectic behavior. In the British literature, rheopectic behavior is called anti-thixotropic behavior.

Thixotropic Fluids

These are fluids that lose viscosity over time at a constant shear rate. Dispersing agents for cements tend to make them thixotropic. Viscosity is recovered after the cessation of shear. In some systems, the time it takes to recover is so short that a sheared sample can not be poured out of a container before it gets too thick to flow.

1.4 Kinematic Viscosity

The kinematic viscosity, (ν), is derived from gravity driven flow measurements. Usually a glass capillary viscomer is used. Kinematic viscosity can be derived from the shear viscosity (μ) by dividing μ by the fluid density.

$$\nu = \frac{\mu}{\rho}$$

Units are $\frac{area}{time}(\frac{ft^2}{s}, \frac{cm^2}{s}, etc)$.

1.5 Surface Tension

Surface tension, (σ), is a property of a liquid surface. It describes the strength of the surface interactions. The units on surface tension are $\frac{Force}{length}$. Surface tension is the driving force for water beading on a waxy surface and free droplets of liquid assuming a spherical shape. Lowering of surface tension can be accomplished by adding species that tend collect at the surface. This breaks up the interactions between the molecules of the liquid and reduces the strength of the surface. Pure water has a surface tension that approaches $72\frac{dyne}{cm}$. Surface tension can be reduced by adding surface active agents (surfactants) to the liquid.

1.6 Pressure

Pressure is defined as force divided by the area that the force acts over and therefore has units of $\frac{F}{A}$. It can be a result of an applied force (for example pumping) or hydrostatic (weight of a column of fluid). The total pressure is the sum of the applied and hydrostatic pressure.

$$P = P_{applied} + P_{hydrostatic}$$

2 Fluid Statics

Outline

1. Basic Equation of Fluid Statics
2. Pressure - Depth Relationships
   a) Constant density fluids
   b) Ideal gases
3. Pressure
4. Pressure Vessels and Piping
5. Buoyancy
6. Pressure Measurement
   a) Manometers
   b) Pressure gages
   c) Pressure transducers
7. Static Pressure Head
8. Acceleration

2.1 Basic Equation of Fluid Statics

The pressure at a point in a static fluid is the same in all directions. What this means that the pressure on the small cube in Figure 2.1 is the same on each face. Since depth increases in the downward z-direction, the zero point will be at the top of the fluid.

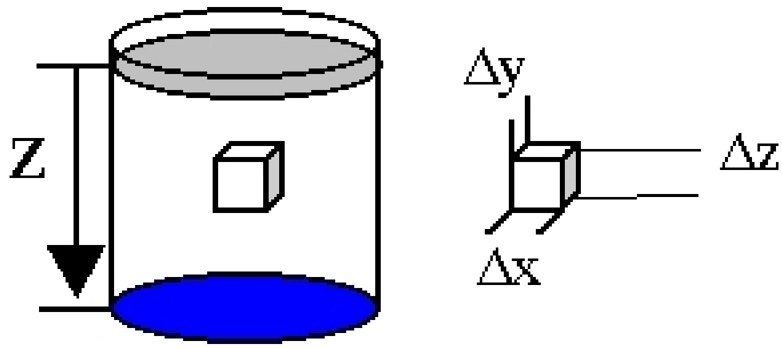


Figure 2.1: Hydrostatic pressure

A simple force balance and some algebra yields

$$\frac{dP}{dz} = -\rho\, g = -\gamma \tag{2.1}$$

For angles, (θ), that deviate from vertical the equation becomes

$$\frac{dP}{dz} = -\rho\, g \cos\theta = -\gamma \cos\theta \tag{2.2}$$

2.2 Pressure - Depth Relationships

2.2.1 Constant Density Fluids

To establish the relationship between pressure and depth for a constant density fluid, the pressure - position relationship must be separated and integrated

$$\frac{dP}{dz} = -\rho\, g \Longrightarrow \int dP = \int -\rho\, g\, dz$$

$$\boxed{P = -\rho g z = -\gamma z} \tag{2.3}$$

Equation 2.3 is one of the most important and useful equations in fluid statics.

Example 2.1. A $5000 ft$ well is filled with a drilling mud that has a specific weight of $11.2 \frac{lb_f}{ft^3}$. What is the pressure in *psig* at the bottom of the well?

$$h = \Delta z = 0\, ft - 5000\, ft = -5000\, ft$$

$$P = -h\gamma$$

$$P = -h \frac{12\, in}{ft} \Big| \frac{11.2\, lb_f}{gal} \Big| \frac{7.48\, gal}{t^3} \Big| \frac{ft^2}{144\, in^2}$$

$$\boxed{P = \frac{-(-5000\, ft)}{} \Big| \frac{12\, in}{ft} \Big| \frac{11.2\, lb_f}{gal} \Big| \frac{7.48\, gal}{t^3} \Big| \frac{ft^2}{144\, in^2} = 2909\, psig}$$

2.2.2 Variable Density Fluids

Most fluids are relatively incompressible, but gases are not. This means that the density increases with depth so we cannot use the specific weight as a constant in determining the pressure. If we assume an ideal gas, then the density is given by

$$\rho = \frac{PM}{RT}$$

where M is the molecular weight, T is the absolute temperature, and R is the gas constant in appropriate units.
Replacing ρ in the differental equation we get

$$\frac{dP}{dz} = -\rho g \quad \Longrightarrow \quad \frac{dP}{dz} = -\frac{PM}{RT} g \tag{2.4}$$

$$\frac{dp}{P} = -\frac{gM}{RT}\,dz \quad \Longrightarrow \quad \int_{P_1}^{P_2} \frac{1}{P}\,dP = -\frac{gM}{RT}\,dz \tag{2.5}$$

Which yields

$$\ln(\frac{P_2}{P_1}) = -\frac{gM}{RT}\,(z_2 - z_1) \quad \Longrightarrow \quad P_2 = P_1\,e^{-\frac{gM}{RT}\Delta z} \tag{2.6}$$

Because the it is possible for the temperature to be non-uniform and gases are poor conductors of heat, the correct equation to use is

$$P_2 = P_1\left(1 - \frac{k-1}{k} \cdot \frac{gM\Delta z}{RT_1}\right)^{\frac{k}{k-1}} \tag{2.7}$$

$$T_2 = T_1\left(1 - \frac{k-1}{k} \cdot \frac{gM\Delta z}{RT}\right) \tag{2.8}$$

Where k is the ratio of the heat capacities.

2.3 Pressure Forces

Pressure acting on a surface exerts a force. This force can be calculated from

$$dF = PdA \quad \Longrightarrow \quad \int dF = \int PdA \quad \Longrightarrow \quad F = \int PdA$$

For constant pressure

$$F = PA \tag{2.9}$$

Example 2.2. Using the result of the previous example ($P = 2900\,psi$), what is the force acting on the bottom of the hole if the hole is eight inches in diameter?

$$F = PA$$

$$F = 2909\,\frac{lb_f}{in^2}\,\frac{\pi}{4}\,(8\,in)^2 = 877334\,lb_f$$

Example 2.3. You are the proud possessor of a 200 ft tall water tower. At the base of the tower (1 $foot$ above the ground) is a seven foot tall by two foot wide access panel that is held in place by bolts. What is the force on the plate?

$$F = -(7\,ft)(2\,ft)\int_{199\,ft}^{192\,ft} \gamma_w\,dh$$

$$F = 170789\,lb_f$$

Since the panel is rectangular, the average pressure can be used.

$$P = -\frac{192\,ft + 199\,ft}{2}\,\gamma_w = 12199\,\frac{lb_f}{ft^2}$$

$$F = \frac{12199\,lb_f}{ft^2}\Big|\frac{7\,ft}{}\Big|\frac{2\,ft}{} = 170789\,lb_f$$

2.4 Buoyancy

Remembering that in a liquid at rest the pressure at a point is the same in all directions, a body submerged in the liquid will experience a force on the top and bottom of the body. Because the body is finite in length, the force on the bottom will be greater than the force on the top. The easy way to think about the magnitude of the buoyant force is that it is equal to the weight of the liquid displaced. It is probably better to consider it as a force balance. To see how this works we will use a length of pipe suspended in a well.

Example 2.4.
A 5000 ft well is filled with a drilling mud that has a specific weight of 11.2 $\frac{lb_f}{gal}$. A 1000 ft piece of pipe ($od = 8\,in$, $id = 7\,in$) is submerged at the surface with only the top of the pipe exposed. What is the buoyant force on the pipe?

$$h = \Delta z = 0\,ft - 1000\,ft$$

$$P_b = -\frac{-1000\,ft}{}\Big|\frac{12\,in}{ft}\Big|\frac{11.2\,lb_f}{gal}\Big|\frac{7.48\,gal}{ft^3}\Big|\frac{ft^3}{(12\,in)^3} = 582\frac{lb_f}{in^2}$$

$$F_B = P_b\,\frac{\pi}{4}\left((8\,in)^2 - (7\,in)^2\right) = 5940\,lb_f$$

What happens if the bottom of the pipe is at 3500 ft.

$$P_b = -\frac{-3500\,ft}{}\Big|\frac{12\,in}{ft}\Big|\frac{11.2\,lb_f}{gal}\Big|\frac{7.48\,gal}{ft^3}\Big|\frac{ft^3}{(12\,in)^3} = 2036\frac{lb_f}{in^2}$$

$$P_t = -\frac{-2500\,ft}{}\Big|\frac{12\,in}{ft}\Big|\frac{11.2\,lb_f}{gal}\Big|\frac{7.48\,gal}{ft^3}\Big|\frac{ft^3}{(12\,in)^3} = 1454\frac{lb_f}{in^2}$$

$$F_B = (P_b - P_t)\,\frac{\pi}{4}\left((8\,in)^2 - (7\,in)^2\right) = 5940\,lb_f$$

2.5 Pressure Measurement

2.5.1 Manometers

There are numerous ways that pressure can be measured. Two examples are shown in Figures 2.2 and 2.3.

Most books, your's included, present a manometer equation, which for simple manometers is easy to use. For a manometer having more than one fluid and a complex topography, the following three rules are

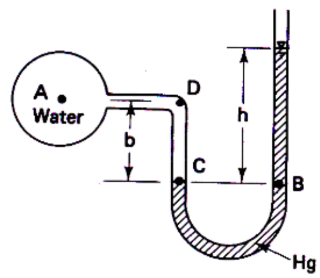


Figure 2.2: Open-end manometer.

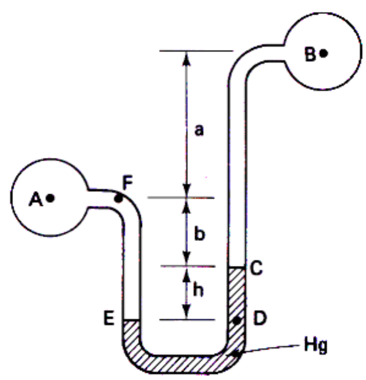


Figure 2.3: Differential manometer.

far easier to use.

Starting at one end of the manometer

1. Work through the system considering only one fluid at a time
2. Add pressure differences as you proceed down through a fluid from the starting point (or subtract when working upwards).
3. Move horizontally through a fluid without change in the pressure

Set the resulting equation equal to the pressure at the other end of the manometer.

Applying these rules is simple. In Figure 2.4, the tank and tube form a simple manometer. To find the pressure at point **A**, we start by writing the pressure at point **A** as P_A. Appling rule 3, we move from point **A** to point **B**. Since both points are at the same level, the pressures are equal.

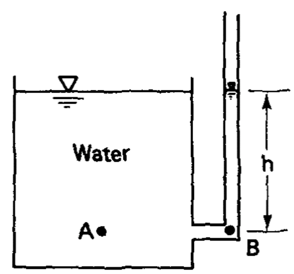


Figure 2.4: A tank with a standpipe (simple manometer).

Rule 2 says to subtract going up so $h\gamma$ is subtracted from P_A and since there is no flow $P_A - h\gamma$ must be equal to the pressure at the top of the tube (P_{atm}). This produces the equation

$$P_A - h\gamma = P_{atm}$$

For more complicated manometers containing more fluids and more legs, just add terms to the equation to describe each one.

Example 2.5. Looking at the manometer in Figure 2.3 we can, by applying the rules, write an equation for the manometer.
Begin by designating the pressure at point A as P_A.
Applying rule **3**, move horizontally to point F where the pressure is equal to P_A.
Applying rule **2**, add the contribution to the pressure from the column of fluid $(F \rightarrow E)$, $(b+h)\,\gamma_w$.

$$P_A + (b+h)\,\gamma_w$$

Applying rule **3**, move horizontally from E to D
Applying rule **2**, subtract the contribution from the mercury column, $-h\,\gamma_{Hg}$

$$P_A + (b+h)\,\gamma_w - h\,\gamma_{Hg}$$

Applying rule **2**, subtract the contribution from the water column $C \rightarrow B$, $-(b+a)\,\gamma_w$

$$P_A + (b+h)\,\gamma_w - h\,\gamma_{Hg} - (b+a)\,\gamma_w$$

Since there is no flow, set the equation = to the pressure at B, P_B.

$$P_A + (b+h)\,\gamma_w - h\,\gamma_{Hg} - (b+a)\,\gamma_w = P_B$$

This becomes the manometer equation. Some algebra allows us to determine the pressure difference between points A and B, ΔP

2.5.2 Bourdon Tube

Bourdon tube pressure gages (Figure 2.5) consist of a closed-end metallic tube that is curved in to almost a circle. Application of pressure causes the tube to ’straighten’ a small amount and the motion is converted into a rotation of the needle on a dial. These gages are simple and rugged, so they are used in a wide range of applications in a plant environment.

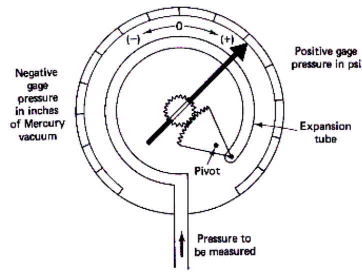


Figure 2.5: Bourdon tube pressure gage

2.5.3 Pressure Transducers

A transducer is a device that turns a mechanical signal into an electrical signal or an electrical signal into a mechanical response. There are a number of ways to accomplish this.

1. Strain gage
2. Capacatance
3. Variable reluctance
4. Optical

2.6 Accelerated Rigid-Body Motion

Changes in motion bring about acceleration or deceleration. A gasoline truck accelerating or decelerating will cause the fluid in the tanks to move to one end of the tank or the other. The major change in the analysis of pressure is that the force balance becomes equal to the *mass times accleration* rather than zero.

$$\frac{dP}{dz} = -\rho\left(g + \frac{d^2 z}{dt^2}\right) \tag{2.10}$$

This equation can be integrated for constant density fluids to yield

$$P_2 - P_1 = -\rho\left(g + \frac{d^2 z}{dt^2}\right)(z_2 - z_1) \tag{2.11}$$

For gauge pressure

$$P = -\rho\, h \left(g + \frac{d^2 z}{dt^2} \right) = -h\,\gamma + \left(\rho\, \frac{d^2 z}{dt^2} \right) \tag{2.12}$$

Equation (2.12) has the static pressure term $(-h\,\gamma)$ and an acceleration term, $\left(\rho\, \frac{d^2 z}{dt^2}\right)$. Evaluating the static pressure term is relatively easy, the acceleration term can be a little more difficult.

3 Balance Equations

Almost all of engineering comes down to balances – mass, momentum, and/or energy. We have already seen in developing the manometer equation that it is nothing more than a pressure balance. In the beginning chemical engineering course, the general balance equation was presented. Evaluating the various components of the equation is the challenge in various aspects of chemical engineering.

$$Flow\,in - Flow\,out + Generation - Consumption = Rate\,of\,Accumulation$$

Simple mass balances without chemical reactions leave out the generation and consumption terms.

$$\dot{m}_{in} - \dot{m}_{out} = Rate\,of\,Accumulation$$

So, if the rate of accumulation is zero (steady-state), we have

$$\dot{m}_{in} = \dot{m}_{out}$$

Based on balance arguments, the following important relationships can be derived.

$$\boxed{Q = vA} \tag{3.1}$$

In Equation 3.1, v is the velocity and A is the cross-sectional area. For constant density fluids

$$Q_{in} = Q_{out}$$

so

$$\boxed{v_{in}A_{in} = v_{out}A_{out}} \tag{3.2}$$

3.0.1 Equation of Continuity

Because mass must be conserved

$$\dot{m}_1 = \dot{m}_2$$

and

$$\boxed{\dot{m} = \rho\, v A} \tag{3.3}$$

the continuity equation becomes

$$\boxed{\rho_{in}\, v_{in} A_{in} = \rho_{out}\, v_{out} A_{out}} \tag{3.4}$$

or

$$\boxed{\sum \rho\, v A = 0} \tag{3.5}$$

The equation of continuity can also be written in terms of the specific weight.

Example 3.1. Water flows into a cylindrical tank through pipe 1 at the rate of 20 ft/s and leaves through pipes 2 and 3 at the rate of 8 ft/s and 10 ft/s, respectively. At 4 we have an open air vent. The following are the inside diameters of the pipes: $D_1 = 3\, in$, $D_2 = 2\, in$, $D_3 = 2.5\, in$, and $D_4 = 2\, in$

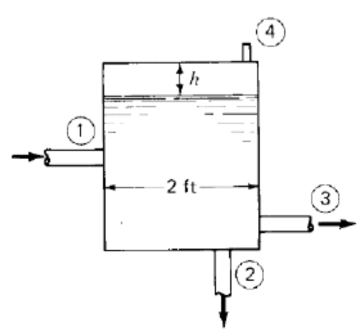


Figure 3.1: Example: Flow into and out of a tank.

Solution
1. Calculate the mass flow rates in using $\dot{m} = \rho\, v A$

$$\dot{m}_1 - \dot{m}_2 - \dot{m}_3 = \dot{m}_{tank}$$

$$\rho\, v_1 A_1 - \rho\, v_2 A_2 - \rho\, v_3 A_3 = \rho\, v_{tank} A_{tank}$$

It is clear that the value of ρ is the same on both sides of the equation so it can be eliminated. Since $area = \frac{\pi}{4}d^2$ and the area term is on both sides of the equation, the $\frac{\pi}{4}$ term cancels.

$$\frac{20\,ft}{s}\left(\frac{3\,in}{}|\frac{ft}{12\,in}\right)^2 - \frac{8\,ft}{s}\left(\frac{2\,in}{}|\frac{ft}{12\,in}\right)^2 - \frac{10\,ft}{s}\left(\frac{2.5\,in}{}|\frac{ft}{12\,in}\right)^2 = v_{tank}\,(12\,ft)^2$$

Solving for v_{tank} yields

$$v_{tank} = 0.148\frac{ft}{s}$$

Later it will be necessary to think of the v_{tank} term as the differential $\left(\frac{dh_{tank}}{dt}\right)$. Using Equation 3.2 and assuming that the air in the tank is incompressible

$$v_{tank} A_{tank} = v_4 A_4$$

$$\frac{0.148\,ft}{s}|\frac{\pi}{4}|\left(\frac{2\,ft}{}\right)^2 = v_4|\frac{\pi}{4}|\left(\frac{2in}{}|\frac{ft}{12\,in}\right)^2$$

Solving for v_4 gives

$$\boxed{v_4 = 21.38\,\frac{ft}{s}}$$

3.1 Control Volume

The control volume is an imaginary sub-volume in a flow system through which flows occur. It can be denoted with a dotted or dashed line enclosing the volume (Figure 3.2).
This may seem like an unnecessary complication, but with another simple addition the control volume concept becomes quite useful. If

Figure 3.2: Control volume

the area through which the flow occurs is declared to be a vector quantity directed outward from the control volume $\left(\vec{A} = A\vec{n}\right)$, it can be combined with the velocity vector using the dot product to produce the proper sign for the flow when Equation 3.5 is used $\left(\sum \rho\, v \cdot \vec{A} = 0\right)$. The dot product produces $v \cdot \vec{A} \Longrightarrow vA\cos(\theta)$, where θ is the angle between the area and velocity vector. Since the cosine of zero is one and the cosine of 180 is (-1), the direction of flow is set relative to the area vector. It is important to draw the control volume such that the velocity vector is perpendicular to the control volume surface (Figure 3.3).

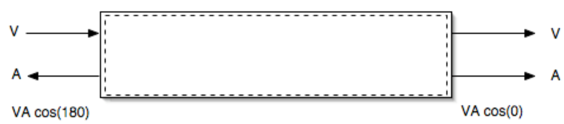


Figure 3.3: Control volume with area and velocity vectors

The previous example would become

$$\rho\, v_1 A_1 \cos 180 + \rho\, v_2 A_2 \cos 0 + \rho\, v_3\, v_3 \cos 0 + \rho\, v_4 A_4 \cos\theta = 0$$

Where the sign of the final answer would determine the direction of flow. Plugging in numbers we find that $A_4 \cos\theta$ is positive so the flow is outward.

3.2 Fluid Velocity in a Confined Region

There are two ways to consider the velocity of a fluid in a flow channel. The average velocity is probably the most straightforward. It is simply the volumetric flow rate divided by the area of the conduit.

$$v_{ave} = \frac{Q}{A} \tag{3.6}$$

In reality, unless the fluid is flowing very fast or very slow, there is a significant velocity distribution that for flow of a Newtonian fluid in a pipe is parabolic (Figure 3.4). One of the basic assumptions of fluid mechanics is that the fluid velocity at the wall is zero i.e. no slip. With the velocity equal to zero at the wall, the fluid velocity is at a maximum at or near the centerline of the flow (depends upon the geometry of the flow channel).

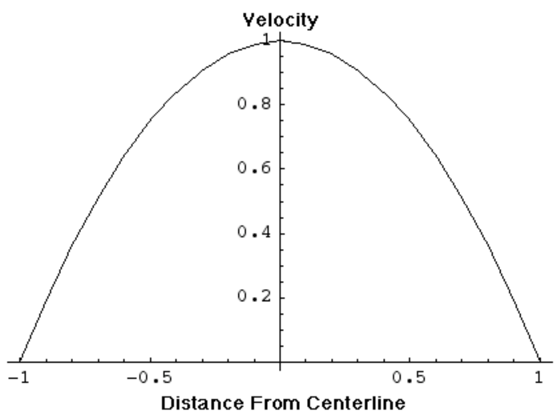


Figure 3.4: Velocity distribution for a Newtonian fluid in laminar flow.

3.2.1 Flow Regimes

Depending upon the fluid velocity, the fluid can flow in different flow regimes. At low flow rates, the fluid particles move in a uniform fashion along the channel with only a small difference in the velocity between the wall and the centerline. As the velocity increases, the flow remains uniform, but the difference between the velocity at the centerline and the wall becomes large. This flow regime is termed the laminar region or simply laminar flow. At some point the velocity increases enough that the flow becomes chaotic and the uniformity in the flow disappears (termed turbulent flow). The velocity profile becomes flat and the flow is strongly mixed. Flow regimes can be separated by the value of the Reynolds number. This is a dimensionless group that represents the ratio of the inertial forces to the viscous forces and is given by

$$N_{Re} = \frac{\rho\, d\, v}{\mu}$$

where d is the characteristic diameter of the flow channel, ρ is the fluid density, v is the velocity, and μ is the fluid viscosity.

3.2.2 Plug or Creeping Flow

At extremely low flow rates, the fluid is normally in plug or creeping flow. This flow regime has a flat velocity profile with much smaller difference between the velocity of the fluid flowing near the wall and at the centerline (Figure 3.5). Experimentally, it is a difficult flow regime to maintain. This flow regime is mathematically simpler than the other two flow regimes so it has been the subject of a great deal of study. Reynolds numbers are typically less than one for plug or creeping flow.

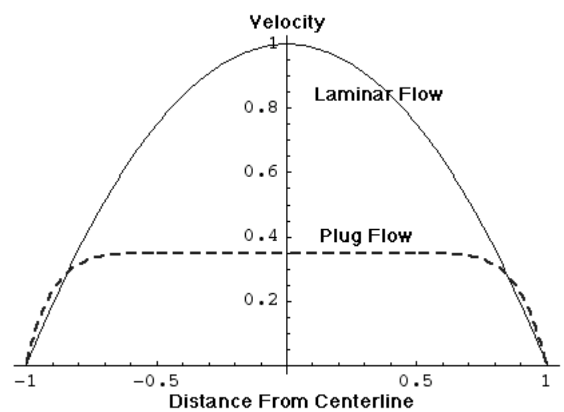


Figure 3.5: Velocity distribution for a Newtonian fluid in plug flow and laminar flow.

3.2.3 Laminar Flow

As velocity increases above the plug flow limit, a velocity profile similar to that labeled as laminar in Figure 3.5. The Reynolds number range for laminar flow is one to 2,100 (it may be somewhat higher in non-Newtonian fluids). In steady (time derivative equal to zero), uniform flow a fluid particle will flow along the same streamline. This means that there is no mixing of fluid across streamlines in laminar flow. In extreme cases where vibrations have been excluded, the laminar flow regime can extend to a Reynolds number of 60,000 or so.

3.2.4 Turbulent Flow

Turbulent flow profiles exhibit a flat velocity profile with a thin laminar layer near the wall. The motion of fluid particles in turbulent flow is chaotic and the flow is strongly mixing. If you have ever watched smoke rise from a cigarette, you should have noticed that it rises in a narrow channel (laminar flow) and then breaks into a plume when the velocity becomes high enough to reach the turbulent flow regime. Almost all flow in water pipes, air ducts, and pipelines is turbulent.

3.3 Unsteady-State Mass Balances

There are a number of real world problems that are not steady-state problems. What this means is that the rate of accumulation term in the equation below is not zero but it is equal to $\frac{dm}{dt}$.

$$\dot{m}_{in} - \dot{m}_{out} = Rate\,of\,Accumulation$$

So the equation becomes

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm}{dt}$$

Example 3.7 in de Nevers needs some explanation. In this example, a tank that is initially full of air is being pumped down with a vacuum pump. At a constant volumetric flow rate ($Q = \frac{1\,ft^3}{min}$), how long does it take to reduce the pressure to 0.0001 atm? As with any balance problem, the best place to start is by writing the balance equation

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm}{dt}$$

In this system, there is no flow in so the $\dot{m}_{in}$ term is zero and the equation reduces to

$$\frac{dm}{dt} = -\dot{m}_{out}$$

At this point, use the definition of density and rearrange the equation to find the mass of gas in the system using the density and system volume

$$m_{system} = \rho_{system} \forall_{system}$$

Taking the time derivative of both sides of the equation yields

$$\left(\frac{dm}{dt}\right)_{system} = \forall_{system} \frac{d\rho_{system}}{dt} + \rho_{system} \frac{d\forall_{system}}{dt}$$

but the system volume is constant so the last term is equal to zero and the equation becomes

$$\left(\frac{dm}{dt}\right)_{system} = \forall_{system} \frac{d\rho_{system}}{dt}$$

Assuming that the gas is not compressible, $m = \rho \forall$, however, we are interested in the mass flow rate $\dot{m}$ so Q is used instead of $\forall$, yielding

$$\dot{m}_{out} = Q_{out}\,\rho_{out}$$

Since Q is constant $\rho_{out} = \rho_{system}$ which leads to

$$\dot{m}_{out} = Q_{out}\,\rho_{system}$$

We can now write the differential equation for the system

$$\forall_{system} \frac{d\rho_{system}}{dt} = -Q_{out}\,\rho_{system}$$

Separating the variables yields

$$\frac{d\rho_{system}}{\rho_{system}} = -\frac{Q_{out}}{\forall_{system}}\,dt$$

Integrating the equation yields

$$\ln\left(\frac{\rho_{system_f}}{\rho_{system_i}}\right) = \frac{Q_{out}}{\forall_{system}}\Delta t$$

Remembering that

$$\rho_{gas} = \frac{P\overline{M}}{RT}$$

this equation can be solved for Δt.

3.4 Summary

Mass balances are extremely important in fluid flow. Missing mass usually means that there is a leak. Using the equation of continuity seems difficult at first and dealing with the dot product of velocity and area $\left(v \cdot \vec{A}\right)$ appears to add an unnecessary complication, learning how to set up problems using this formulation will help to minimize mistakes in the long run. In Chemical Engineering, unsteady state mass balances are important. It is worth spending some time to try to understand them.

4 The First Law of Thermodynamics

In the last chapter, we covered mass balances. As a prelude to the Bernoulli equation in the next chapter, we need to discuss energy balances. The energy balance, like the mass balance, does not have creation or destruction terms and so the general balance equation reduces to

$$flow\,in - flow\,out = Accumulation$$

There are several forms of energy that we must deal with in fluid mechanics.

1. Kinetic energy – energy due to motion
2. Potential energy – energy due to position
3. Internal energy – energy of the atoms and molecules that make up a system
4. Surface energy – energy due to surface interactions

The other forms of energy are generally not useful in fluid mechanics, but may find specialized uses in areas such as magentohydrodynamics.

4.1 Energy Transfer

Energy may be transfered through heat flow or work. Heat flows from hot bodies to colder bodies. This form of transfer forms the basis for thermodynamics. Work is defined as

$$Work = \int force \cdot d(distance) = \int F\,dx$$

Work can also be done by rotating shafts.

4.2 Energy Balance

Returning to the general balance equation and remembering that energy cannot be created or destroyed we get

$$flow\,in - flow\,out = Accumulation$$

Your book uses lower case symbols for specific values as opposed to the carrot or hat above the symbol. For instance

$$u \equiv \hat{U}$$

It also used upper case symbols for the non specific values as shown below

$$U = m\,u$$

Energy may enter a system in several ways. It may be carried in across the flow boundary by mater entering or leaving the system. Heat transfer to or from the system is also a conduit for energy transfer. Work may done on or by the system can also serve as a transfer mechanism. Figure 4.1 shows an example of each of the mechanisms.

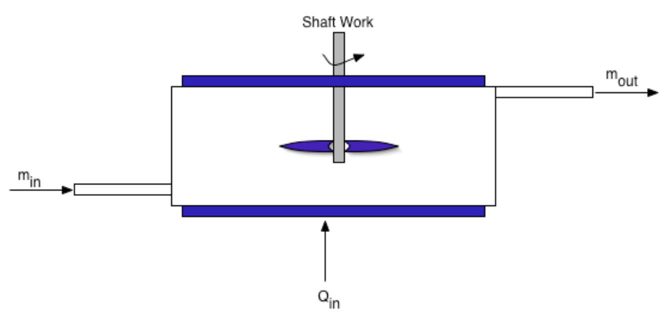


Figure 4.1: Energy input from flow, shaft work, and heat transfer

So the overall energy balance for the system becomes

$$d[m(u+pe+ke)]_{sys} = [(u+pe+ke)_{in}dm_{in} + dQ_{in} + dW_{in} - [(u+pe+ke)_{out}dm_{out} + dQ_{out} +$$

This equation can be simplified by defining

$$dQ = dQ_{in} - dQ_{out}$$

and

$$dW = dW_{in} - dW_{out}$$

yielding

$$d[m(u + pe + ke)]_{sys} = [(u + pe + ke)_{in} dm_{in} - [(u + pe + ke)_{out} dm_{out} + dQ + dW \tag{4.1}$$

4.2.1 Sign Conventions

Sign conventions have always been a bit confusing. It seems difficult to imagine that one can just arbitrarily pick a sign for flow into or out of the system and get the correct answer. But, it works as long as you pick a sign and stick with it. The sign convention in your book is

- Work done on the system (flowing in) is positive
- Heat transfered to the system (flowing in) is positive

This results in

$$dQ + dW = 0$$

4.2.2 Potential Energy

The specific form of the potential energy equation simplifies to

$$pe = gz$$

If one looks at a flowing system, the specific form of the potential energy equation becomes

$$\Delta pe = pe_{in} - pe_{out} = g\left(z_{in} - z_{out}\right)$$

To get the total potential energy change we need to multiply the specific quantity by the mass of the system.

$$\Delta PE = m\Delta pe$$

4.2.3 Kinetic Energy

The kinetic energy term reduces to

$$ke = \frac{v^2}{2}$$

or

$$KE = mke = m\frac{v^2}{2}$$

For systems with velocity change

$$\Delta KE = m\left(\frac{v_{in}^2}{2} - \frac{v_{out}^2}{2}\right)$$

4.3 Internal Energy

For a closed system of constant mass with no kinetic or potential energy, the internal energy can be written as

$$mdu = dQ + dW$$

Integration yields

$$U = mu = Q + W + constant$$

The constant is based on some reference point. Typically, it is water at the triple point where u is set equal to zero.

4.4 Work

Work on a compressible system is given by

$$dW = Fdx = PAdx = -Pd\cancel{V}$$

Since the volume decreases, the sign is negative. Injection work is the work required to inject a mass across system boundaries. Rather than deal with the injection work and the internal energy, the enthalpy $(h = u + P\cancel{V})$ is used. So that the energy balance equation becomes

$$\begin{aligned} d\left[m\left(h+gz+\frac{v^2}{2}\right)\right]_{sys} = \\ & \left(h+gz+\frac{v^2}{2}\right)_{in} dm_{in} \\ & -\left(h+gz+\frac{v^2}{2}\right)_{out} dm_{out} \\ & + dQ + dW \end{aligned} \tag{4.2}$$

4.5 Summary

The first law can be used to analyze energy flow into and out of a system. Equation 4.2 provides the basis for the Bernoulli equation discussed in the next chapter.

5 Bernoulli Equation

One of the most useful equations in fluid mechanics is the Bernoulli equation (BE). Your book presents it in the differential form

$$\Delta\left(\frac{P}{\rho}+gz+\frac{v^2}{2}\right)=0$$

where ΔP is $P_{out}-P_{in}$. Looking at the units we find

$$\frac{F}{l^2}|\frac{l^3}{mass}+\frac{l}{t^2}|\frac{l}{-}+\frac{l^2}{t^2}|\frac{}{2}$$

In order to straighten this mess out, it is necessary to divide ρ byg_c or to express force in terms of $mass \times acceleration$. Either way yields units of $\frac{l^2}{t^2}$.

$$\frac{F}{l^2}|\frac{l^3}{mass}|\frac{mass\,l}{t^2\,F}+\frac{l}{t^2}|\frac{l}{-}+\frac{l^2}{t^2}|\frac{}{2}$$

There is a better way to think about the Bernoulli equation. Normally, the BE is written between two points in the flow.

$$\frac{P_1}{\rho}+g\,z_1+\frac{v_1^2}{2}=\frac{P_2}{\rho}+g\,z_2+\frac{v_2^2}{2} \tag{5.1}$$

Neglecting friction and work and dividing both sides by g the Bernoulli equation may also be written as

$$\frac{P_1}{\gamma}+z_1+\frac{v_1^2}{2\,g}=\frac{P_2}{\gamma}+z_2+\frac{v_2^2}{2\,g} \tag{5.2}$$

This form has the advantage of having units of length or head. Otherwise, ρ in Equation 5.1 has to be divided by g_c to make the units work out.

5.1 Applying the Bernoulli Equation

In applying the BE, it is best to pick the two points (1 and 2) to zero out as many terms as possible. For instance picking the two points along the same horizontal line eliminates z_1 and z_2. As an example, consider a tank full of water that is draining from an opening near the bottom of the tank (Figure 5.1)

By selecting points 1 and 2 (point 2 is just at the exit) as shown in

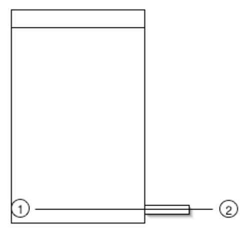


Figure 5.1: Bernoulli equation tank draining example

Figure 5.1, z_1 and z_2 are equal and cancel each other out. V_1 is small relative to V_2 and can be ignored and P_2 is equal to P_{atm} so for gauge calculations it is equal to zero. We are left with P_1 and v_2 to evaluate. The position of the points can be selected in a slightly different manner to make the problem even easier as shown in Figure 5.2.

In this case, P_1, z_1, V_1, and P_2 are all equal to zero. This leaves z_2 and

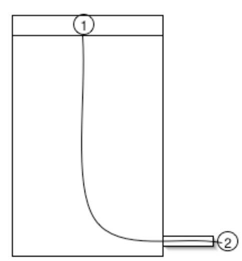


Figure 5.2: Bernoulli equation tank draining example

v_2 to be evaluated. The quantity, z_2, is just the distance from the surface of the liquid to the midpoint of the flow and is equal to $(-h)$. Evaluating the velocity, v_2

$$0+0+0=0+(-h)+\frac{v_2^2}{2g}$$

Solving for v_2

$$\boxed{v_2=\sqrt{2gh}} \tag{5.3}$$

Which is known as Torricelli's equation.

Example 5.1. **Example:** In the venturi meter shown below, the $D=40cm$ and $d=10cm$. What is the water discharge (flow rate) in the system, if the pressure at A is $100kPa$, $P_{atm}=101.325kPa$, the water temperature is $10°C$, and cavitation has just started at point 2?

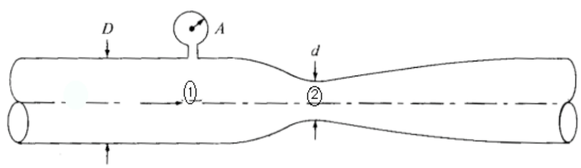


Figure 5.3: Venturi tube.

Solution:

- The pressure at point 1 is given by
  $P_1=100{,}000Pa+101{,}325Pa$
- The pressure at point 2 is set by the vapor pressure of water at $10°C$. $P_2=1230Pa$
- Since the points are along the same horizontal line, $z_1=z_2$.

Writing the Bernoulli equation between points 1 and 2

$$\frac{P_1}{\gamma}+z_1+\frac{v_1^2}{2g}=\frac{P_2}{\gamma}+z_2+\frac{v_2^2}{2g}$$

canceling the elevation terms and rearranging the equation we get

$$\left(\frac{P_1}{\gamma}-\frac{P_2}{\gamma}\right)=\frac{v_2^2}{2g}-\frac{v_1^2}{2g}$$

Now we must express v_2 in terms of v_1.

$$v_1A_1=v_2A_2 \Longrightarrow v_2=\frac{v_1A_1}{A_2}$$

and solve the BE equation for V_1

$$\left(\frac{P_1}{\gamma}-\frac{P_2}{\gamma}\right)=\frac{\left(v_1\frac{A_1}{A_2}\right)^2}{2g}-\frac{v_1^2}{2g}$$

$$\frac{v_1^2}{2g}=\frac{\left(\frac{P_1}{\gamma}-\frac{P_2}{\gamma}\right)}{\left(\frac{A_1}{A_2}\right)^2}$$

$$V_1=\sqrt{2g\frac{\left(\frac{P_1}{\gamma}-\frac{P_2}{\gamma}\right)}{\left(\frac{A_1}{A_2}\right)^2}}$$

substituting in the numeric values yields

$$v_1=1.25$$

Since we know $Q=vA$

$$Q=0.157\frac{m^3}{s}$$

It should be noted that cavitation is just boiling. When the pressure equals the vapor pressure of the liquid at the specified temperature, boiling occurs. So when a problem specifies cavitation, the pressure at the point of cavitation is fixed by the vapor pressure of the liquid at the specified temperature.

5.2 Bernoulli Equation With Friction

Up to now we have neglected frictional forces. These forces increase pressure drop and in some cases cause heating. The Bernoulli equation can be modified to include friction by adding a friction term $(\mathscr{F})$ to the right side of the equation. Note: In deNevers the friction term has a negative sign because he has rearranged the equation to yield the Δ values to be final state minus initial state i.e. $P_2 - P_1$ so don't be confused. The Bernoulli equation becomes

$$\frac{P_1}{\gamma} + z_1 + \frac{v_1^2}{2g} = \frac{P_2}{\gamma} + z_2 + \frac{v_2^2}{2g} + \frac{\mathscr{F}}{g} \tag{5.4}$$

The friction term $\mathscr{F}$ takes the form of a constant times $\left(\frac{v^2}{2g}\right)$. Frictional forces arise from simple flow, flow through valves, elbows, orifices, etc. Values for the constant can be obtained from tables or from the friction factor charts.

Example 5.2. Fluid flowing through the pipe and elbow shown below exhibits frictional losses both in the pipe and the elbow. The pipe friction will be treated in Chapter 6 and will be neglected in this example.

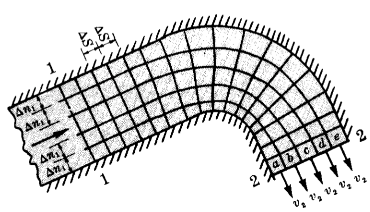


Figure 5.4: Flow through an elbow.

The Bernoulli equation for this flow system becomes

$$\frac{P_1}{\gamma} + z_1 + \frac{v_1^2}{2g} = \frac{P_2}{\gamma} + z_2 + \frac{v_2^2}{2g} + \frac{\mathscr{F}}{g}$$

If the elbow is in plane perpendicular to the page, $z_1 = z_2$. Since there is no change in velocity, $v_1 = v_2$. So the Bernoulli equation reduces to

$$\frac{P_1}{\gamma} = \frac{P_2}{\gamma} + \frac{\mathscr{F}}{g}$$

which simplifies to

$$\frac{P_1 - P_2}{\gamma} g = \mathscr{F}_b$$

Measuring ΔP then allows us to calculate the friction associated with the elbow. If the flow rate is known then the constant, K_b in the equation below can be evaluated.

$$\mathscr{F} = K\frac{v^2}{2}$$

5.3 Gas Flows

Gasses at low velocities can be treated as incompressible fluids. The pressure drop must also be low or the change in volumetric flow rate and thus, the fluid velocity will cause problems. This means that the BE can be used in many situations that are of interest to Chemical Engineers.

When gases flow through a venturi meter, the velocity increases as the passage gets smaller and decreases as passage returns to the original size. Figure 5.5 is an idealized venturi tube with sharp changes in the shape of the tube. Normally, the changes are more gradual, without abrupt changes (like those shown in the figure). This means that the pressure will decrease as the velocity increases on the left side of the restriction and recover on the right side of the restriction. This can be seen in Figure 5.6. Pressure recovery on the downstream side of a restriction is a common occurrence. To understand this, we need to look at a cross-section of the flow in the venturi tube at several points At the first increment, Δx, the pressure, $P1$, must be greater than $P2$

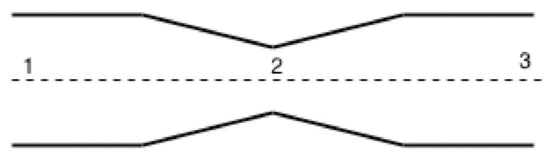


Figure 5.5: Idealized venturi tube.

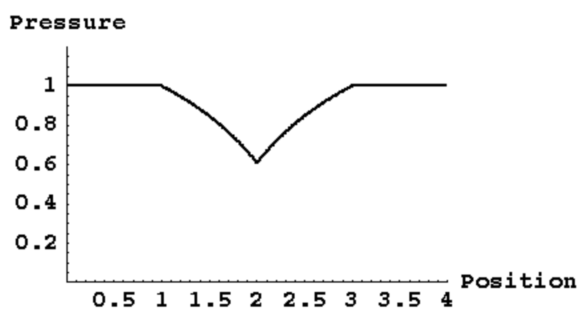


Figure 5.6: Pressure profile in a venturi tube.

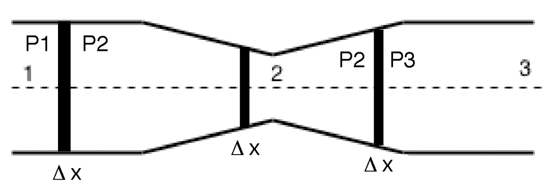


Figure 5.7: Flow in a venturi tube.

for flow to occur. The increment near 2 in Figure 5.7 is characterized by an increase in velocity that necessitates an increase in the pressure differential across the increment. In the third increment, the velocity is decreasing and $P3$ must be greater than $P2$ to provide the force necessary to decelerate the flow.

5.4 Non-Flow Work

In the previous chapter, it was shown that the flow work could be buried in the enthalpy. This left all of the non-flow work to be accounted for. Non-flow work is often called shaft work, since in fluid flow it is usually the result of a shaft turning. The Bernoulli equation can be extended to include the work term by adding it to the left side of the equation. Again be careful when comparing Equation 5.5 with the equation in your book.

$$\frac{P_1}{\gamma} + z_1 + \frac{v_1^2}{2g} + \frac{dW_{nf}}{g} = \frac{P_2}{\gamma} + z_2 + \frac{v_2^2}{2g} + \frac{\mathscr{F}}{g} \tag{5.5}$$

The work term consists of all of the non-flow work. It includes work done on the system by stirrers, mixers, pumps, etc. or work done by the system turning turbines, shafts, or other rotating machinery. We will return to non-flow work in Chapter 6.

5.5 Flow Measurement

The Bernoulli equation can be exploited to measure fluid flow rates. Before we explore the different ways that the BE equation can be used, we need to talk about the nature of flow around bodies. For a fixed body in a flow, the fluid will approach the body and flow around it as shown in Figure 5.8. The dark point in the center of the body represents the stagnation point. At this point, the flow has a velocity of zero i.e. the flow is stagnant. Since the fluid velocity at the stagnation point is zero, it gives us a point in the system to anchor the BE equation.

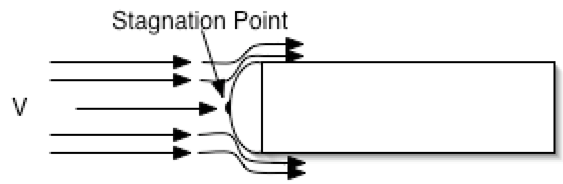


Figure 5.8: Stagnation point.

5.5.1 Pitot Tube

A pitot tube is a simple device used to measure flow rates. It works for both liquid and gas flows. The device, in its simplest form, consists of a small diameter hollow tube bent into the shape of a L. Usually the upstream opening is smaller than the diameter of the tube. The height, above the center-line, of the fluid in the in the vertical leg of the tube is related to the velocity of the fluid in the flow. An example is shown in Figure 5.9.

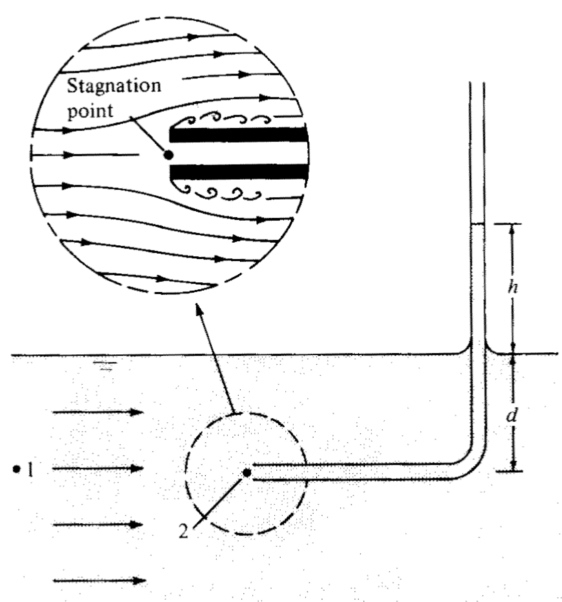


Figure 5.9: Pitot tube

Writing the Bernoulli equation between points 1 and 2

$$\frac{P_1}{\gamma} + z_1 + \frac{v_1^2}{2g} = \frac{P_2}{\gamma} + z_2 + \frac{v_2^2}{2g}$$

and zeroing out z_1, z_2, and v_2 we get

$$\frac{P_1}{\gamma} + \frac{v_2}{2g} = \frac{P_2}{\gamma}$$

Assigning values to the various parameters

$$P_1 = P_{atm} + b\gamma$$

$$P_2 = P_{atm} + (b+h)\,\gamma$$

Substituting into the Bernoulli equation, neglecting friction, and solving for v_1 we obtain

$$v_1 = \sqrt{2gh} \tag{5.6}$$

5.5.2 Static Pitot Tube

Some books refer to the first example of a pitot tube as a stagnation tube and the static pitot tube (pitot static tube) as a pitot tube. We will use deNevers nomenclature. The difference between the pitot tube and the static pitot tube is the
small opening on the side of the submerged part of the tube. Unlike the stagnation tube (Figure 5.9), a direct measurement of ΔP is possible. Using the Bernoulli equation and neglecting friction,

$$\frac{P_1}{\gamma} + z_1 + \frac{v_1^2}{2g} = \frac{P_2}{\gamma} + z_2 + \frac{v_2^2}{2g} \Longrightarrow \frac{P_1}{\gamma} + \frac{v_1^2}{2g} = \frac{P_2}{\gamma}$$

solving for v_1

$$v_1 = \sqrt{2g\frac{\Delta P}{\gamma}} = \sqrt{2\frac{\Delta P}{\rho}} \tag{5.7}$$

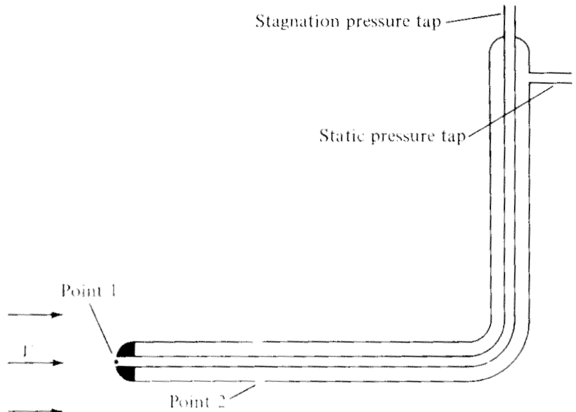


Figure 5.10: Pitot static tube.

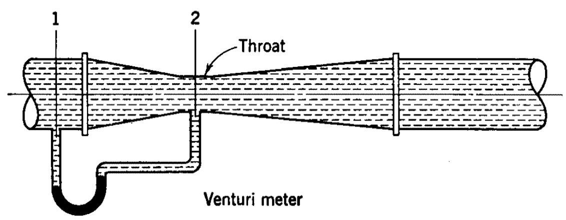


Figure 5.11: Venturi meter

5.5.3 Venturi Meter

We have previously discussed the venturi tube, but not as a method of flow measurement. It has the advantage of no moving parts and if it is designed properly friction can be ignored.
The Bernoulli equation can be written between points 1 and 2. For horizontally mounted tubes, the elevation terms can be zeroed out.

$$\frac{P_1}{\gamma} + z_1 + \frac{v_1^2}{2g} = \frac{P_2}{\gamma} + z_2 + \frac{v_2^2}{2g}$$

If the diameter of the tube at 1 is d_1 and the diameter at 2 is d_2, the velocity at point 1 can be expressed in terms of the velocity at point 2 by solving $v_1 A_1 = v_2 A_2$ for v_1. Furthermore, the area ratio can be expressed as the ratio of the squares of the diameters. This leads to the expression of the Bernoulli equation

$$\frac{P_1}{\gamma} + \frac{\left(v_2 \frac{d_2^2}{d_1^2}\right)^2}{2g} = \frac{P_2}{\gamma} + \frac{v_2^2}{2g}$$

This equation can be solved for v_2

$$v_2 = \sqrt{\frac{2\left(P_1 - P_2\right)}{\rho\left(1 - \frac{d_2^2}{d_1^2}\right)}} \tag{5.8}$$

Application of the Bernoulli equation to the venturi tube in this manner ignores friction and other important factors. This results in a small error in the measured flow rates. To correct for this error, the meters are calibrated and a calibration constant provided. This results in the corrected venturi tube equation 5.9.

$$v_2 = C_v \sqrt{\frac{2\left(P_1 - P_2\right)}{\rho\left(1 - \frac{d_2^2}{d_1^2}\right)}} \tag{5.9}$$

5.5.4 Orifice Meter

Orifice meters are, in some respect, an extreme example of a venturi meter. The orifice meter has an abrupt change in diameter rather than the more gradual change in the venturi meter. An orifice meter has a plate, usually removable, with a small hole machined in the center. The upstream side of the hole is flat, while the downstream side is cone-shaped. One of the common mistakes in installing an orifice meter is reversing the plate. This results in erroneous measurements. Gas measurements made with orifice meters are analyzed using a complex formula (Figure 5.12) that goes far beyond the Bernoulli equation. The full orifice meter equation even includes the local gravitational acceleration!

Needless to say the full equation is seldom used, but it is easy to see why a great deal of accuracy is needed. Consider the case of a natural gas well flowing into a pipeline. If the well is flowing $5MMcfd$ (5 million cubic feet/day) and there is a 0.1 percent error in the meter reading, the gain or loss in gas sales is about thirty dollars per day or

$$Q_h = C' \sqrt{h_w P_f}$$

where Q_h = quantity rate of flow at base conditions in cu ft/hr
h_w = differential pressure in inches of water
P_f = absolute static pressure, psia
C' = orifice flow constant

$$C' = F_b \cdot F_r \cdot Y \cdot F_{pb} \cdot F_{tb} \cdot F_{tf} \cdot F_{gr} \cdot F_{pv} \cdot F_a \cdot F_{am} \cdot F_{wl} \cdot F_{wt} \cdot F_{pwl} \cdot F_{hgm} \cdot F_{hgt}$$

and F_b = basic orifice factor
F_r = Reynolds number factor
Y = expansion factor
F_{pb} = pressure base factor
F_{tb} = temperature base factor
F_{tf} = flowing temperature factor
F_{gr} = specific gravity factor
F_{pv} = supercompressibility factor
F_a = orifice thermal expansion factor
F_{am} = correction for air over the water in the water manometer during the differential instrument calibration
F_{wl} = local gravitational correction for water column calibration standard
F_{wt} = water weight correction (temperature) for water column calibration standard
F_{pwl} = local gravitational correction for dead weight tester static pressure standard
F_{hgm} = manometer factor, correction for gas column in mercury manometers
F_{hgt} = mercury manometer temperature factor

Figure 5.12: Full orifice meter equation

eleven thousand dollars per year at the current price of natural gas. For a field that produces a billion cubic feet per day (there are fields that produce this much) the difference is in excess of two million dollars!

5.5.5 Rotameters

A rotameter is a simple device that is used when a high degree of accuracy is not needed. The gas or liquid flow is used to levitate a ball of known diameter and density. Usually the rotameter is calibrated so that a measured position of the ball corresponds to a flow rate. Care must be exercised when using rotameters. They are most accurate when the ball is located in about the middle two-thirds of the meter.

Measurement error increases at the ends.

5.6 Unsteady Flows

There are a number of situations where unsteady flows are encountered. The simplest is the draining of a tank where the flow rate varies with the height of the fluid in the tank. The Torricelli equation (5.3) really only gives you a snapshot of the flow rate in time. If the tank diameter is large the flow rate changes slowly, but if it is small the flow rate change can be significant.

For a simple tank (Figure 5.13) Since Torrecelli's equation gives the

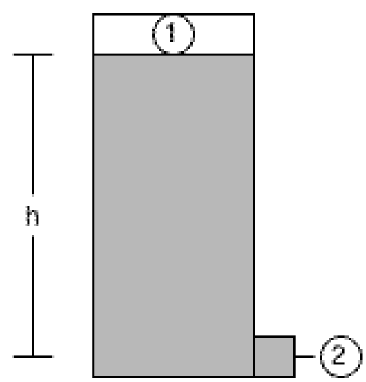


Figure 5.13: Unsteady flow example.

instantaneous flow rate we can write the velocity at 2 as

$$v_2 = \sqrt{2gh}$$

and if we solve $v_1 A_1 = v_2 A_2$ for v_2, we now have

$$v_1 \frac{A_1}{A_2} = \sqrt{2gh}$$

v_1 is only the rate at which the height is dropping $\left(\frac{dh}{dt}\right)$. Rearrange and solve for $\left(\frac{dh}{dt}\right)$ yields the differential equation

$$-\frac{dh}{dt} = \frac{A_2}{A_1}\sqrt{2g,h} \tag{5.10}$$

This equation can be solved by separating the variables and integrating or by using the ordinary differential equation solver in MatLab.

$$\int_{h_1}^{h_2} \frac{1}{\sqrt{h}}\, dh = \frac{A_2}{A_1} \sqrt{2g} \int_{t_1}^{t_2} dt \tag{5.11}$$

When the tank has parallel sides this problem is relatively easy, however, if the tank is cone shape the area of the surface becomes a function of height. Finding the functional form of the diameter is first problem and integrating it is the second. For the cone-shaped tank shown in Figure 5.14 as the level in the tank drops, the cross-sectional area A_1 decreases.

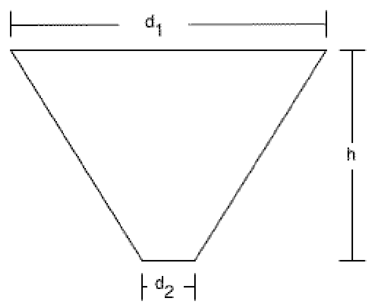


Figure 5.14: Cone shaped tank.

The area of a cone as a function of height can be derived using the two diameters and the height. Since we are dealing with a truncated cone, we have to break the problem down into two parts (Figure 5.15). First the area between the two horizontal lines is constant and equal to $\frac{\pi}{4}d_2^2$. Next we have to determine angle,θ. We know the value of $h_{initial}$ and we can derive the length of the opposite side of the triangle (o) by

$$o = \frac{d_1 - d_2}{2}$$

Using h and o, $\tan\theta$ can be calculated

$$\tan\theta = \frac{o}{h}$$

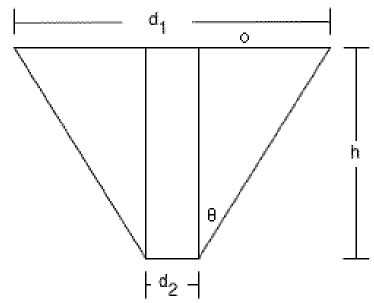


Figure 5.15: Cone shaped tank.

Combining the numeric value for $\tan\theta$ with any value for the height yields the length of o at the selected height. The diameter, d_1 at any height between h_0 and h can now be expressed by

$$d_1(h) = 2(\tan\theta)(h) + d_2$$

Integrating Equation 5.11 becomes a bit more difficult because A_1 is now a function of height. The A_1 terms now looks like

$$A_1 = \frac{\pi}{4}\left(2(\tan\theta)(h) + d_2\right)^2$$

and the integral is

$$-\int_{h_1}^{h_2} \frac{\left(2(\tan\theta)(h) + d_2\right)^2}{\sqrt{h}}\,dh = \frac{A_2}{\frac{\pi}{4}}\sqrt{2g}\int_{t_1}^{t_2} dt \qquad (5.12)$$

This integral is somewhat more difficult to integrate than the one in Equation 5.11, but there are a number of calculators or computer programs (MatLab, Maple, or Mathematica) that can do the integral.

Example 5.3. **Example:** In Figure 5.16, the diameter of the top of the tank is $2\,ft$ and the diameter of the bottom is $0.5\,ft$. The tank is $15\,ft$ high. Derive an equation to calculate the time needed to empty the tank given

$$\frac{-dh}{\sqrt{h}} = \frac{A_2}{A_1}\sqrt{2\,g}\,dt$$

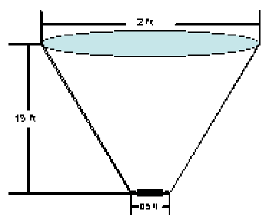


Figure 5.16: Conical tank example.

Expressing A_1 as

$$A_1 = \frac{\pi}{4}\left(2\left(\tan\theta\right)(h) + d_2\right)^2$$

The integral that must be solved is

$$-\int_0^{15}\frac{A_1}{\sqrt{h}}\,dh = A_2\sqrt{2g}\int_{t_1}^{t_2}dt$$

Assigning values to the various parameters

$$A_2 = \frac{\pi}{4}\left(0.5\,ft\right)^2 = 0.196\,ft^2$$

$$\tan\theta = \frac{0}{h} = \frac{\frac{d_2-d_1}{2}}{h} = \frac{\frac{2\,ft-0.5\,ft}{2}}{15\,ft} = 0.05$$

$$A_1 = \frac{\pi}{4}\left(2\,(0.05)\,(h) + 0.5\,ft\right)^2 = (0.1\,h + 0.5)^2$$

The integral is now

$$-\int_0^{15} \frac{(0.1\,h + 0.5)^2}{\sqrt{h}}\, dh = 0.196\, ft^2 \sqrt{2\left(32.2\frac{ft}{s^2}\right)} \int_{t_1}^{t_2} dt$$

$$\Delta t = -\frac{\int_0^{15} \frac{(0.1\,h+0.5)^2}{\sqrt{h}}\, dh}{0.196\, ft^2 \sqrt{2\left(32.2\frac{ft}{s^2}\right)}}$$

Analytical solution of this equation is difficult, but it is easy to evaluate numerically. The answer is

$$\Delta t = 4.6\, s$$

5.7 Summary

The Bernoulli equation describes the relationship between velocity and pressure. It is useful in a wide variety of flow problems. Care must be taken in choosing the start and end points. In general, the points should be selected to minimize the number of variables that must be evaluated. In the next chapter we will see how the Bernoulli equation can be extended to more complex flow systems where friction is important and work is done on or by the system.

6 Fluid Friction in Steady One-Dimensional Flow

Unlike the systems that were covered in the last chapter, real fluids do not flow without frictional losses. Flow through pipes, valves, expansions, contractions, bends or into or out of tanks are just a few examples of sources of friction. One of the results of friction is pressure drop.

7 Momentum Balance

7.1 Newton's Laws

1. A body at rest remains at rest and a body in motion remains in motion at the same velocity in a straight path when the net force acting on it is zero.

2. The acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass.

3. When a body exerts a force on a second body, the second body exerts an equal and opposite force on the first. This is the so called reaction force.

Expressing the second law in equation form

$$F = ma = m\frac{dv}{dt} = \frac{d\,(mv)}{dt} \tag{7.1}$$

The mv term is called the linear momentum or simply the momentum. Looking at the second law equation (7.1), it can also be thought of as *the rate of change of the momentum of a body is equal to the net force acting on the body*. For our purposes, this is probably a better way of thinking about the second law. It is important to remember that the force, acceleration, velocity, and momentum are vectors, i.e. they have both a magnitude and direction.

7.2 Control Volumes

Most of the work in working momentum problems is in the bookkeeping. Properly selecting a control volume and control surface makes the process easier. If the subject of the analysis is moving, then it is usually better to let the control volume move. For a fixed flow system, the control volume should be fixed. For example, the system shown in 7.1 should have a control volume that encompasses the nozzle and the plate. For a jet exiting a moving airplane, the velocity

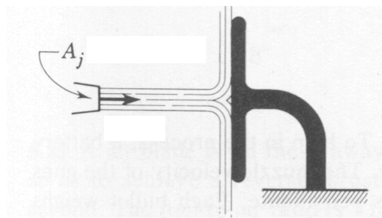


Figure 7.1: Force on a plate exerted by a jet of fluid impinging on the plate at a 90° angle.

of interest is the relative velocity. An example is shown in Figure 7.2. In this case, the velocity is given by

$$v_x = v_{jet} - v_{CV}$$

Where the jet is moving in the negative *x-direction* while the control

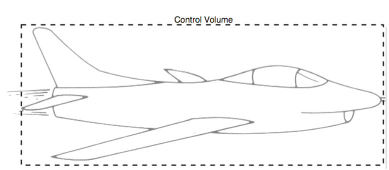


Figure 7.2: Moving airplane

volume is moving in the positive *x-direction*. This yields a negative overall velocity $\left(v = -v_{jet} - v_{CV}\right)$. An example of a deformable control volume is the control volume inside a piston chamber in a motor (Figure 7.3). The control volume expands and contracts as the piston moves up and down.

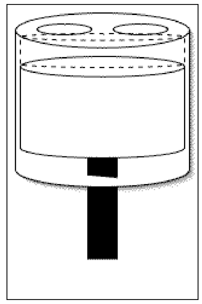


Figure 7.3: Deformable control volume.

7.3 Forces on a Control Volume

There are two types of forces that act on a control volume: *Body* and *Surface*. The body forces act throughout the whole body while the surface forces act on the control surface. Body forces can be a gravity, electric, or magnetic field. Surface forces of interest in fluid flow are pressure and viscous forces. Normally, we can ignore all of the body forces except gravity.

$$\sum F = \sum F_{body} + \sum F_{surface} \tag{7.2}$$

Gravitational force on a fluid element

$$dF_{gravity} = \rho\, g\, d\forall \tag{7.3}$$

Total force acting on a control volume

$$\sum F_{body} = \int_{CV} \rho\, g\, d\forall = m_{CV}\, g \tag{7.4}$$

Surface force acting on a small surface element

$$d\vec{F}_{surface} = \sigma \cdot \vec{n} dA \tag{7.5}$$

Where $\vec{n}$ is an outwardly pointing vector normal to the surface dA and σ is the stress on the surface, dA (Figure 7.4).
The total force on a body is

$$\sum \vec{F} = \sum \vec{F}_{body} + \sum \vec{F}_{surface} = \int_{CV} \rho\, g\, d\forall + \int_{CV} \sigma\, \vec{n}\, dA \tag{7.6}$$

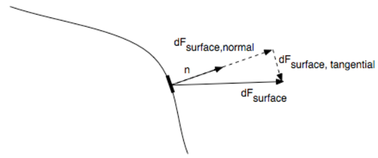


Figure 7.4: Surface forces acting on a small area.

7.4 Steady Flow

In steady flow, the equation reduces to

$$\sum \vec{F} = \sum_{out} \dot{m}\vec{v} - \sum_{in} \dot{m}\vec{v}$$

Which reduces to

$$\sum F = \dot{m}\left(\vec{v}_{out} - \vec{v}_{in}\right) \tag{7.7}$$

Example: Liquid Jet Striking a Flat Plate

Water strikes a flat plate at a rate of $10 kg/s$ with a velocity of $20 m/s$ and exits in all directions in the plane of the plate. What is the force necessary to hold the plate in place?

Solution: All of the entering velocity is in the positive *x-direction* and none of the fluid exits in the *x-direction*. This means that v_{2x} is equal to zero.

$$\sum F = \dot{m}\left(\vec{v}_{2x} - \vec{v}_{1x}\right)$$

reduces to

$$-F_x = 0 - \dot{m}v_1$$

$$F_x = \dot{m}v_1$$

$$F_x = \frac{10kg}{s} \Big| \frac{20m}{s} \Big| \underbrace{\frac{N \cdot s^2}{kg \cdot m}}_{g_c} = \boxed{200 \text{ N}}$$

Example: Liquid Jet Striking a Curved Plate

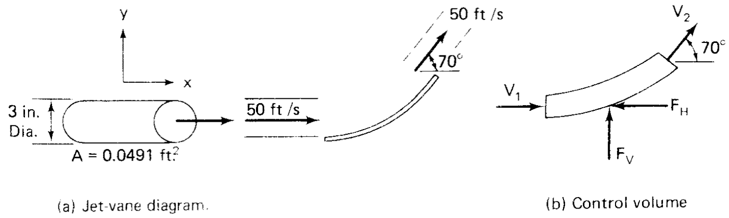


Figure 7.5: Force on a curved vane

A 3-inch diameter jet with a velocity of 50 ft/s is deflected through an angle of 70° when it hits a stationary vane, determine the horizontal and vertical components of the force of the water on the vane.
Solution: Neglecting the weight of the fluid, height changes in the fluid, the weight of vane and assuming no frictional losses, the velocity of the exiting liquid jet must be the same as the entering jet. This does not preclude changes in direction or cross-sectional shape. Vector analysis must be used to determine the *x* and *y* velocities.

$$\sum F = \sum_{out} \dot{m}\vec{v} - \sum_{in} \dot{m}\vec{v}$$

The *x-component* of the velocity is given by

$$v_{2,x} = v_2 \cos\theta = \frac{50\,ft}{s} | \cos\left(\frac{70^\circ}{} | \frac{2\pi}{360^\circ}\right) = 17.1\frac{ft}{s}$$

The *y-component* of the velocity is given by

$$v_{2,y} = v_2 \sin\theta = \frac{50\,ft}{s} | \sin\left(\frac{70^\circ}{} | \frac{2\pi}{360^\circ}\right) = 46.98\frac{ft}{s}$$

Calculating the mass flow rate

$$\dot{m} = \rho\, q \quad where \quad q = v_1 A$$

$$q = \frac{50 ft}{s} | \frac{0.0491 ft^2}{} = 2.455\frac{ft^3}{s}$$

$$\dot{m} = \frac{62.4\, lb_m}{ft^3} | \frac{2.455 ft^3}{s} = 153.19\frac{lb_m}{s}$$

The reaction force in the *x-direction* is given by

$$F_x = \dot{m}\left(v_{2,x} - v_{1,x}\right)$$

$$F_x = \frac{153.19\,lb_m}{s}|\frac{1}{g_c}|\left(17.1\frac{ft}{s} - 50\frac{ft}{s}\right) = \boxed{\text{-156.5 } lb_f}$$

The reaction force in the *y-direction* is given by

$$F_y = \dot{m}\left(v_{2,y} - v_{1,y}\right)$$

$$F_y = \frac{153.19\,lb_m}{s}|\frac{1}{g_c}|\left(\frac{46.98ft}{s} - 0\frac{ft}{s}\right) = \boxed{223.5\ lb_f}$$

If we want the force of the fluids acting on the plate, the signs are reversed.

___________________________ Now we will complicate the problem by having the vane move with a velocity in the *x-direction* with a velocity of $15 ft/s$.

___________________________ **Example: Liquid Jet Striking a Moving Curved Plate**

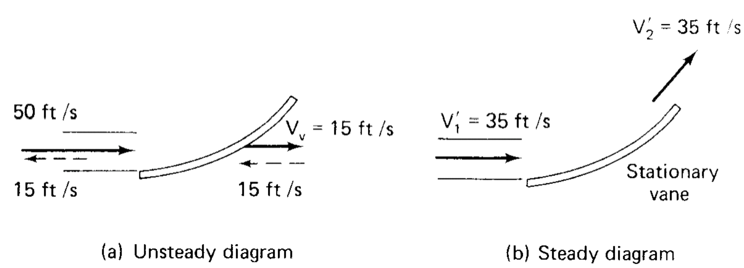


Figure 7.6: Force on a moving curved vane

Using the relative velocity in place of v_1

$$v_{2,x} = v_2 \cos\theta = \frac{35\,ft}{s}|\cos\left(\frac{70^\circ}{}|\frac{2\pi}{360^\circ}\right) = 11.97\frac{ft}{s}$$

$$v_{2,y} = v_2 \sin\theta = \frac{35\,ft}{s}|\sin\left(\frac{70^\circ}{}|\frac{2\pi}{360^\circ}\right) = 32.89\frac{ft}{s}$$

$$F_x = \frac{153.19\,lb_m}{s}|\frac{1}{g_c}|\left(11.97\frac{ft}{s}35\frac{ft}{s}\right) = \boxed{\text{-76.7 } lb_f}$$

$$F_y = \frac{153.19\, lb_m}{s} | \frac{1}{g_c} | \left(\frac{32.89 ft}{s} - 0\frac{ft}{s} \right) = \boxed{109.5\ lb_f}$$

Since the vane is moving away from the jet, the forces will decrease as the jet lengthens (the distance between the jet and the vane increases).

There is another way to think about the momentum equation. If we write the equation as

$$\sum F = \sum \rho\, v v \cdot A \tag{7.8}$$

This really is not that different. Remember that $\dot{m} = \rho\, v A$ and by taking the dot product between the velocity and the area vector (it always points outward from the control surface) we can keep the signs straight. Reworking the first part of the last example we get the following.

Example: Force on a Curved Vane

$$\sum F = \sum \rho\, v v \cdot A$$

The *x-component* of the velocity is given by

$$v_{2,x} = v_2 \cos\theta = \frac{50\, ft}{s} | \cos\left(\frac{70^\circ}{} | \frac{2\pi}{360^\circ} \right) = 17.1 \frac{ft}{s}$$

The *y-component* of the velocity is given by

$$v_{2,y} = v_2 \sin\theta = \frac{50\, ft}{s} | \sin\left(\frac{70^\circ}{} | \frac{2\pi}{360^\circ} \right) = 46.98 \frac{ft}{s}$$

The reactive force in the *x-direction* is given by

$$F_x = \frac{\rho}{g_c} v_{1,x} v_1 \cdot A + \frac{\rho}{g_c} v_{2,x} v_1 \cdot A$$

$$\begin{aligned} F_x &= \frac{62.4\, lb_m}{ft^3} | \frac{lb_f s^2}{32.2 lb_m ft} | \frac{50\, ft}{s} | \frac{50\, ft}{s} | \frac{cos\,(180)}{} | \frac{0.0491\, ft^2}{} \\ &+ \frac{62.4\, lb_m}{ft^3} | \frac{32.2 lb_f s^2}{lb_m ft} | \frac{17.1\, ft}{s} | \frac{50\, ft}{s} \frac{cos\,(0)}{} | \frac{0.0491\, ft^2}{} \\ &= \boxed{\text{-156.5 } lb_f} \end{aligned}$$

The reactive force in the *y-direction* is given by

$$F_y = \frac{\rho}{g_c} v_{1,y} v_1 \cdot A + \frac{\rho}{g_c} v_{2,y} v_1 \cdot A$$

$$\begin{aligned} F_y &= \frac{62.4\,lb_m}{ft^3} \Big| \frac{lb_f s^2}{32.2 lb_m ft} \Big| \frac{0\,ft}{s} \Big| \frac{50\,ft}{s} \Big| \frac{cos\,(180)}{} \Big| \frac{0.0491\,ft^2}{} \\ &+ \frac{62.4\,lb_m}{ft^3} \Big| \frac{32.2 lb_f s^2}{lb_m ft} \Big| \frac{46.98\,ft}{s} \Big| \frac{50\,ft}{s} \frac{cos\,(0)}{} \Big| \frac{0.0491\,ft^2}{} \\ &= \boxed{223.5\ lb_f} \end{aligned}$$

Let us look at a slightly different problem involving flow of a fluid around a bend. The flow is not open to the atmosphere as it was in the previous examples so pressure forces must be accounted for.

Example: Force on a Pipe Bend

Water flows through a 180° vertical reducing bend. The discharge is $0.25 m^3/s$ and the pressure at the center of the inlet of the bend is $150 kPa$. If the bend volume is $0.1 m^3$ and it is assumed that Bernoulli's equation is valid, what is the reaction force required to hold the bend in place. Assume the metal in the bend weighs $500N$.

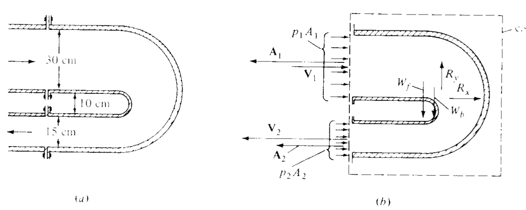


Figure 7.7: Force on a bend.

Solution: Starting with the basic momentum equation, evaluate all of the variables that can be easily evaluated.

$$\sum F = \sum \rho\, v\, v \cdot A$$

$$A_1 = \frac{\pi}{4} d_1^2 = 0.071\, m^3 \qquad A_2 = \frac{\pi}{4} d_2^2 = 0.0177\, m^3$$

$$v_1 = \frac{q}{A_1} = 3.54\,\frac{m}{s} \qquad v_2 = \frac{q}{A_2} = 14.15\,\frac{m}{s}$$

Using the Bernoulli equation to evaluate the missing pressure.

$$\frac{P_1}{\gamma} + z_1 + \frac{v_1^2}{2g} = \frac{P_2}{\gamma} + z_2 + \frac{v_2^2}{2g}$$

$$P_1 = 150\,kPa \qquad z_1 = 0\,m \qquad z_2 = -\left(\frac{d_1}{2} + 0.1\,m + \frac{d_2}{2}\right) = -0.325\,m$$

Solving for P_2

$$P_2 = 59.37\,kPa$$

Returning to the momentum equation we can write

$$F_x + P_1 A_1 + P_2 A_2 = \frac{\rho}{g_c} v_1\, v_1 \cdot A_1 + \frac{\rho}{g_c}\,(-v_2)\, v_2 \cdot A_2$$

Inserting values for the variables and solving for F_x we get

$$F_x = -10.60\,kN - 1.049\,kN - 0.884\,kN - 3.537\,kN = \boxed{-16.07\,kN}$$

The force in the *y-direction* is simply the sum of the weight of the fluid in the bend and the weight of the bend.

$$F_y = W_B + \cancel{V}\rho\,\frac{g}{g_c} = 500\,N + \frac{0.1\,m^3}{}\Big|\frac{1000kg}{m^3}\Big|\frac{9.81m}{s^2}\Big|\frac{N\,s^2}{1\,kg\,m}$$

$$F_y = \boxed{1.48\,kN}$$

Note: If the bend in the previous example is turned around, the *PA* forces have a negative sign in front of them and v_1 is negative rather than v_2. This results in a change in sign, but not magnitude, of F_x. The analysis of a bend of less than 180° is similar, but the angles have to be taken into account when the velocities are calculated.

Example: Flow in a 30° Bend.

A pipe that is 1 m in diameter has a 30° horizontal bend in it, as shown, and carries crude oil $(SG = 0.94)$ at a rate of $2\,\frac{m^3}{s}$. If the pressure in the bend is assumed to be essentially uniform at $75\,kPa\,gage$, if the volume of the bend is $1.2\,m^3$, and if the metal in the bend weighs $4\,kN$,

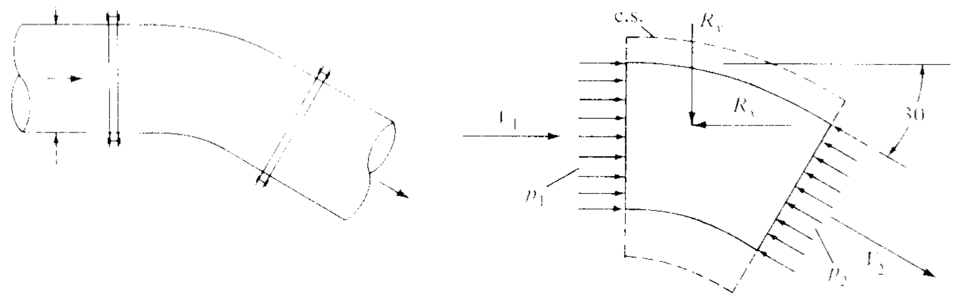


Figure 7.8: Forces on a 30° bend.

what forces must be applied to the bend to hold it in place?
Solution: Starting with the momentum equation

$$\sum F = \sum v \rho\, v \cdot A$$

$$A_1 = A_2 = \frac{\pi}{4} d^2 = 0.785 m^2 \quad v_1 = \frac{q}{A_1} = 2.55\, \frac{m}{s}$$

The velocity at the exit, $v_{2,x}$, is not equal to v_1, but is equal to

$$v_{2,x} = v_1 \cos\left(30\, \frac{2\pi}{360}\right) = 2.205\, \frac{m}{s}$$

So the momentum equation becomes

$$F_x + P_1 A_1 - P_2 A_2 cos\left(30 \frac{2\pi}{360}\right) = \frac{\rho_{oil}}{g_c} v_{1,x} v_1 A_1 \cos(\pi) + \frac{\rho_{oil}}{g_c} v_{2,x} v_2 A_2\, cos(0)$$

Solving for F_x

$$F_x = -59.05\, kN + 51.01\, kN - 4.79\, kN + 4.15\, kN = \boxed{8.64\, kN}$$

There is no pressure force acting in the $y - direction$ on the entrance side of the bend nor is there a $y - component$ of the velocity so

$$F_y + P_2 A_2 \sin\left(30 \frac{2\pi}{360}\right) = \frac{\rho_{oil}}{g_c} v_{2,y} v_2 A_2\, cos(0)$$

$$F_y = 29.45\, kN + 2.39\, kN = \boxed{31.84\, kN}$$

The force in the $z-direction$ results from the weight of the bend and the weight of the fluid enclosed within it.

$$F_z = W_B + W_f = 4\,kN + \frac{1.2m}{}\left|\frac{(0.94)\,1000kg}{m^3}\right|\frac{9.81m}{s^2}\left|\frac{N\,s^2}{1\,kg\,m}\right.$$

$$F_z = \boxed{15.1kN}$$

7.5 Rotational Motion and Angular Momentum

7.5.1 Review

Angular velocity The angular distance traveled per unit time.

$$\begin{aligned}
\omega &= \frac{d\theta}{dt} = \frac{d\,(l/r)}{dt} = \frac{1}{r}\frac{dl}{dt} = \frac{v}{r}\\
\alpha &= \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \frac{1}{r}\frac{dv}{dt} = \frac{a_t}{r}\\
v &= r\omega\\
a_t &= r\alpha
\end{aligned}$$

Where a_t is the acceleration in the tangential direction and v is the linear velocity
For constant rotational motion there must be a force acting tangentially to maintain the angular acceleration. The moment or torque (M) is proportional to the length of the arm that connects the point of action and the center of rotation.

$$M = rF_i = rma_t = mr^2\alpha$$

In integral form

$$M = \int r^2\alpha dm = \underbrace{\left(\int r^2 dm\right)}_{I}\alpha = I\alpha$$

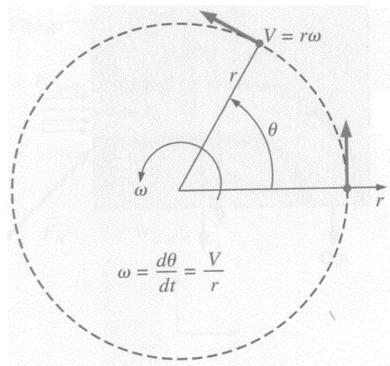


Figure 7.9: Angular velocity

I is the moment of inertia. Remembering that mv is the momentum in a linear system, the moment of momentum or angular momentum for a rotating point mass is given by

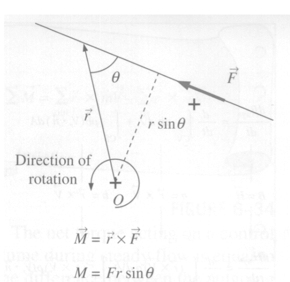
$$H = rmv = r^2 m\omega$$

For a rotating rigid body, the total angular momentum is given by

$$H = \int r^2 \omega dm = \underbrace{\left(\int r^2 dm \right)}_{I} \omega = I\omega$$

Angular velocity $\omega = \frac{2\pi\dot{n}}{60}$ yields rad/s.
In vector form, the momentum force equation becomes

$$\vec{M} = \vec{r} \times \vec{F}$$



Where the magnitude of the moment of force is given by

$$M = Fr \sin\theta$$

The moment of momentum in vector form becomes

$$\vec{H} = \vec{r} \times m\vec{v}$$

For a differential mass dm

$$dm = (\vec{r} \times \vec{v})\, \rho\, d\forall$$

Integration yields

$$\vec{H}_{sys} = \int_{sys} (\vec{r} \times \vec{v})\, \rho\, d\forall$$

For steady flow the net torque can be written as

$$\sum \vec{M} = \sum_{out} \vec{r} \times \dot{m}\vec{v} - \sum_{in} \vec{r} \times \dot{m}\vec{v}$$

______________________________ **Example:** A large lawn sprinkler with four identical arms is to be converted into a turbine to generate electrical power by attaching a generator. Water enters the sprinkler from the base along the axis of rotation at a rate of 20 liters/s and leaves the nozzles in the tangential direction. The sprinkler rotates at a rate of 300 rpm in a horizontal plane. The diameter of each jet is $1cm$, and the normal distance between the axis of ration and the center of each nozzle is $0.6m$. What is the torque on the shaft available to generate electrical power?

Solution: We want to solve the equation

$$\sum M = \sum_{out} r\dot{m}v - \sum_{in} r\dot{m}v$$

The last term is equal to zero since $r_{in} = 0$. Calculate $\dot{m}$

$$\dot{m} = \rho q = \frac{1000\, kg}{m^3} | \frac{0.02\, m^3}{s} = 20 \frac{kg}{s}$$

Calculate the nozzle velocity

$$v_n = \frac{0.02m^3}{s} | \frac{1}{4} | \frac{4}{\pi\, (0.1m)} = 63.66 \frac{m}{s}$$

Calculate the angular velocity

$$\omega = \frac{(2\pi)\, 300\, rev}{min} | \frac{min}{60\, s} = 31.42 \frac{rad}{s}$$

Calculate the tangential velocity

$$v_{nozzle} = r\omega = (0.6m)\left(42\frac{}{s}\right) = 18.85\frac{m}{s}$$

Calculate the relative velocity

$$v_{rel} = v_n - v_r = 63.66 - 18.85 = 44.81\frac{m}{s}$$

Calculate the torque

$$T_{shaft} = r\dot{m}_{total}v_r = \frac{0.6\,m}{}\Big|\frac{20\,kg}{s}\Big|\frac{44.81\,m}{s}\Big|\frac{N\cdot s^2}{kg\cdot m} = \boxed{537.7N\cdot m}$$

Calculating the power

$$Po = \omega T_{shaft} = \frac{31.42}{s}\Big|\frac{537.7N\cdot m}{}\Big|\frac{kW\,s}{1000N\cdot m} = 16.9\,kW$$